

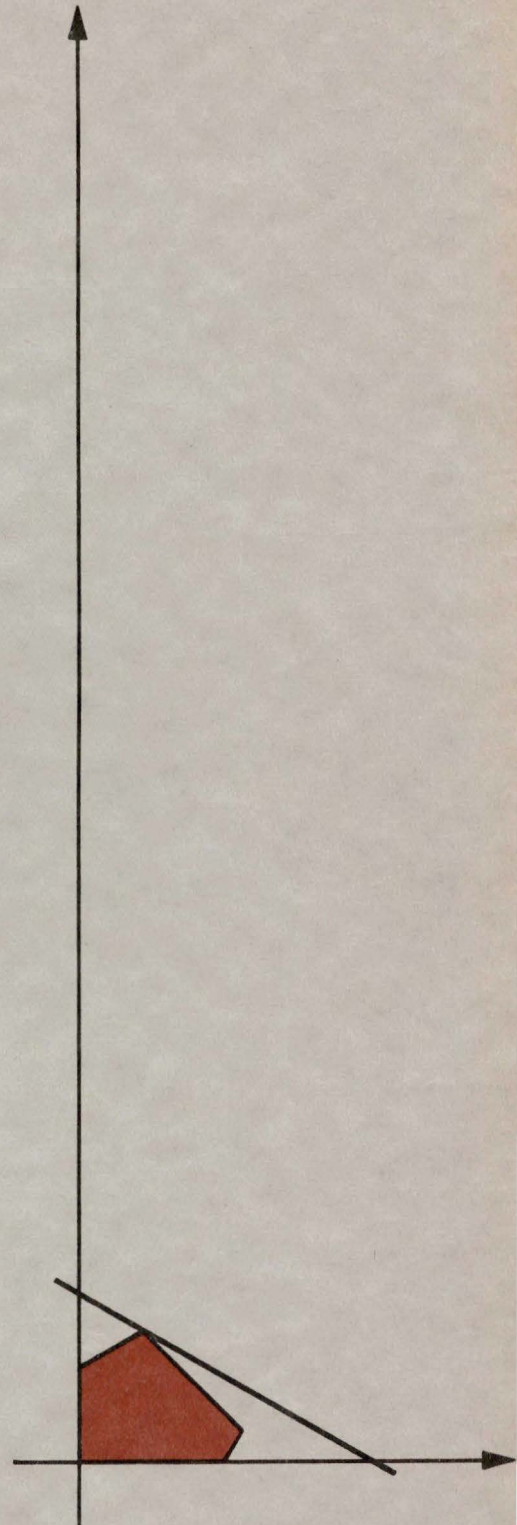
LEAST COST LOCATION OF FIRE SUPPRESSION EQUIPMENT

by

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ABSTRACT

The basic fire-growth model proposed by Parks and Jewell is generalized by considering the time difference between fire detection and the start of suppression activity to be composed of mobilization and travel times. A distance-dependent cost model is formed, considering the probability that the fire is a given distance from a reference point. This model is solved for the location of the suppressing forces that minimizes the expected cost of a fire.

The model is generalized to allow fire growth and fire cost parameters to vary, and the case of varying rates of travel due to hills is considered. Graphical solution methods are presented, and electronic computer techniques can easily be developed. A numerical example is given to illustrate a graphical solution and the use of the model.

It is shown that locating n suppression forces is accomplished by solving n single location problems, under a given set of demarcation points. When the demarcation points are not given, the general problem is formulated and solved for the special case of a constant rate of fire growth.

LEAST COST LOCATION OF FIRE SUPPRESSION EQUIPMENT

I. Introduction

The original models of fire growth and suppression cost^[1] verified that fires should be attacked as soon as possible for minimum suppression and damage costs. One method of reducing delay in attacking fires is to minimize the distance traveled by the suppression forces; this reduces total costs as well. Since not all fires have the same growth rate and cost of area burned, it is preferable to locate the suppression forces closer to areas with high growth rate potential and high land value. Minimizing distance traveled doesn't necessarily optimize fire suppression costs when using optimal sized crews, but minimizing total cost of fire suppression with respect to distance traveled does. Since fire locations are known only in probabilistic terms, total expected cost is to be minimized.

The total cost model of suppressing a forest fire, developed by Parks^[1], is $C = C_F + C_S x + C_T \tau_C + C_B y_C + C_X \tau_C x$ where C_F is the fixed cost of maintaining the fire suppression organization, C_S is the cost per unit of the suppression force, x is the number of units of the force, C_T is the cost proportional to the total time the fire is burning out of control, τ_C is the total time the fire burns out of control, C_B is the value of the burned property, y_C is the burned area, and C_X is a cost proportional to the total number of manhours spent controlling the fire. Parks concludes that C_T can be neglected. We may take $C_T = 0$ for two reasons. First,

consultation with experienced fire control personnel has failed to discover any fire cost items with these dimensions that could be included in this term; and secondly, we will see in Chapter VI that because τ_C^* is invariably quite small, C_T would have to be quite large before $C_T \tau_C^*$ could have effect on total costs." (Reference 1, page 27.) The growth of a fire (in the absence of suppression forces) is assumed to be described by [2]:

$$\frac{dy}{dt} = G_D + Ht$$

$$y(0) = y_D$$

Here G_D , [area/time] is observed rate of growth when the fire is detected, H , [area/time] is the acceleration component of growth, t is time measured from time of detection, and y_D , [area] is the observed area of burn at detection.

Parks and Jewell have combined these two models and analyzed the relationships between suppression costs and fire growth parameters. The important relationships are repeated here for convenience.

$$(1) \quad C = C_F + C_S x + C_X \frac{G_A x}{Ex - H} + C_B \left[y_A + \frac{G_A^2}{2(Ex - H)} \right]$$

$$(2) \quad x^* = G_A \sqrt{\frac{C_B}{2C_S E}} \sqrt{1 + 2C_X + \frac{H}{E}}$$

$$(3) \quad C^* = C_F + \frac{C_S^H}{E} + \frac{C_X^G A}{E} + C_B y_a + 2G_A \sqrt{\frac{C_S C_B}{2E}} \sqrt{1 + 2\epsilon_X}$$

$$(4) \quad \epsilon_X = \frac{C_X^H}{C_B^G A}$$

$$(5) \quad G_A = G_D + H\tau_{A_{acc}}$$

$$(6) \quad y_{A_{acc}} = y_{D_{acc}} + G_D \tau_{A_{acc}} + \frac{H(\tau_{A_{acc}})^2}{2}$$

Generally, $\epsilon_X \ll 1$ so $\sqrt{1 + 2\epsilon_X} \approx 1$. This approximation will be used throughout this paper, and the above formulae reduces to:

$$(2') \quad x^* = G_A \sqrt{\frac{C_B}{2C_S E}} + \frac{H}{E}$$

$$(3') \quad C^* = C_F + \frac{C_S^H}{E} + \frac{C_X^G A}{E} + C_B y_A + 2G_A \sqrt{\frac{C_S C_B}{2E}} .$$

II. The Distance-Dependent Cost Model

The interval of time between detection and attack is due to mobilization and travel; the former is essentially constant and the latter is dependent on the distance to the fire. Hence,

$$(7) \quad \tau_A = K + \frac{u}{r}$$

where K is the mobilization time, u is the distance to the fire, and r is the rate of travel to the fire. If the optimal force starts at a point z units from the origin and the fire is l units from the origin, $u = |z - l|$.

Substituting (7) into (5) and (6)

$$(5a) \quad G_A = G_D + H(K + \frac{u}{r})$$

$$(6a) \quad y_A = y_D + G_D(K + \frac{u}{r}) + \frac{H}{2} (K + \frac{u}{r})^2$$

Substituting equations (5a) and (6a) in equation (2'), we get

the distance-dependent equation,

$$(2a) \quad x^*(u) = \frac{H}{E} + (G_D + HK) \sqrt{C_B/2C_S E} + \left(\frac{H}{r} \sqrt{C_B/2C_S E}\right)u$$

When (5a) and (6a) are substituted into (3a), C^* becomes a function of u and $x^*(u)$. This is the distance dependent cost equation when an optimal suppression force is used, and it will be denoted by $C(u)$.

$$(8) \quad \begin{aligned} C(u) = & \left\{ C_F + C_S[(G_D + HK) + \sqrt{2C_B/C_S E} + H/E] \right. \\ & + \frac{C_X}{E}(G_D + HK) + H \sqrt{2C_S/C_B E} + C_B(G_D K + \frac{1}{2} HK^2) \left. \right\} \\ & + \left\{ C_S H \sqrt{2C_B/C_S E} + \frac{C_X H}{E} + C_B(G_D + HK) \right\} \left(\frac{u}{r}\right) \\ & + \left\{ \frac{C_B H}{2} \right\} \left(\frac{u}{r}\right)^2 \end{aligned}$$

which can be written more concisely as

$$(8a) \quad C(u) = \gamma + \alpha \left(\frac{u}{r}\right) + \frac{\beta}{2} \left(\frac{u}{r}\right)^2$$

where γ represents the constant cost of fire suppression [$\$/$], α the cost of the fire growing [$\$/hr.$], and β is the cost due to the fire's accelerating growth [$\$/hr.^2$].

Now consider a probability distribution function $P(\ell)$ defined on a line L units long. This line can represent a road through a forest that has been stretched out, and $dP(\ell)$ is the conditional

probability that a fire occurs on an interval $[l, l + dl]$ l units from the origin given that a fire is burning on $[0, L]$. To account for differences in road surfaces, distances are measured in "constructive miles," and vehicles travel at the constant rate of r mph. We want to find a point, z^* , at which to locate the suppression forces, such that the expected cost of controlling a fire is minimized. When the suppression forces are located at z , the total expected cost is

$$(9) \quad C(u; z) = \int_0^z C(z - l) dP(l) + \int_z^L C(l - z) dP(l).$$

The necessary condition for a minimum of $C(u; z)$ is found by equating the first partial derivative of $C(u; z)$ with respect to z to zero.

$$\frac{\partial C(u; z)}{\partial z} = 0 \Rightarrow \left(\frac{\alpha}{r} + \frac{\beta z^*}{r^2} \right) P(z^*) - \frac{\beta}{r^2} \int_0^L l dP(l) = \left(\frac{\alpha}{r} - \frac{\beta z^*}{r^2} \right) P^c(z^*)$$

But $\int_0^L l dP(l) = \bar{l}$, the expected location of a fire; and the necessary condition becomes

$$(10) \quad 1 - 2P(z^*) = \frac{\beta}{\alpha r} (z^* - \bar{l}).$$

Notice that if β is very small compared to αr , $P(z^*) \approx \frac{1}{2}$, which means the suppression crews are located so half the fires will occur on either side of them. This will happen if C_B or H are negligible ($\beta = C_B H$) or if $r \gg \beta$; i.e., the rate at which the crews travel to the fire is fast enough to dampen the effect of the acceleration component of fire growth.

A minimizing solution is insured because $\frac{\partial^2}{\partial z^2} C(u; z) \geq 0$, which implies that $C(u; z)$ is a convex function of z^* .

$$\frac{\partial^2}{\partial z^2} C(u; z^*) = \frac{d^2}{dz^2} \left(\frac{\beta}{\alpha r} (z^* - \bar{\ell}) + 2P(z^*) - 1 \right) = \frac{\beta}{\alpha r} + 2p(z^*)$$

Since $p(z)$ is a probability density function, it's always ≥ 0 , and negative values for β , α , and r do not make physical sense;

$\frac{\partial^2}{\partial z^2} C(u; z^*) > 0$. If $\beta = p(z^*) = 0$, thus $\frac{\partial^2}{\partial z^2} C(u; z^*) = 0$; this unlikely event would result in an interval for the optimum location, as will be shown later.

Equation (10) suggests a graphical procedure to evaluate z^* . Since z^* is a particular value of ℓ , the intersection of $1 - 2P(\ell)$ and $\frac{\beta}{\alpha r} (\ell - \bar{\ell})$ both plotted against ℓ , gives the value of z^* . See Figure 1.

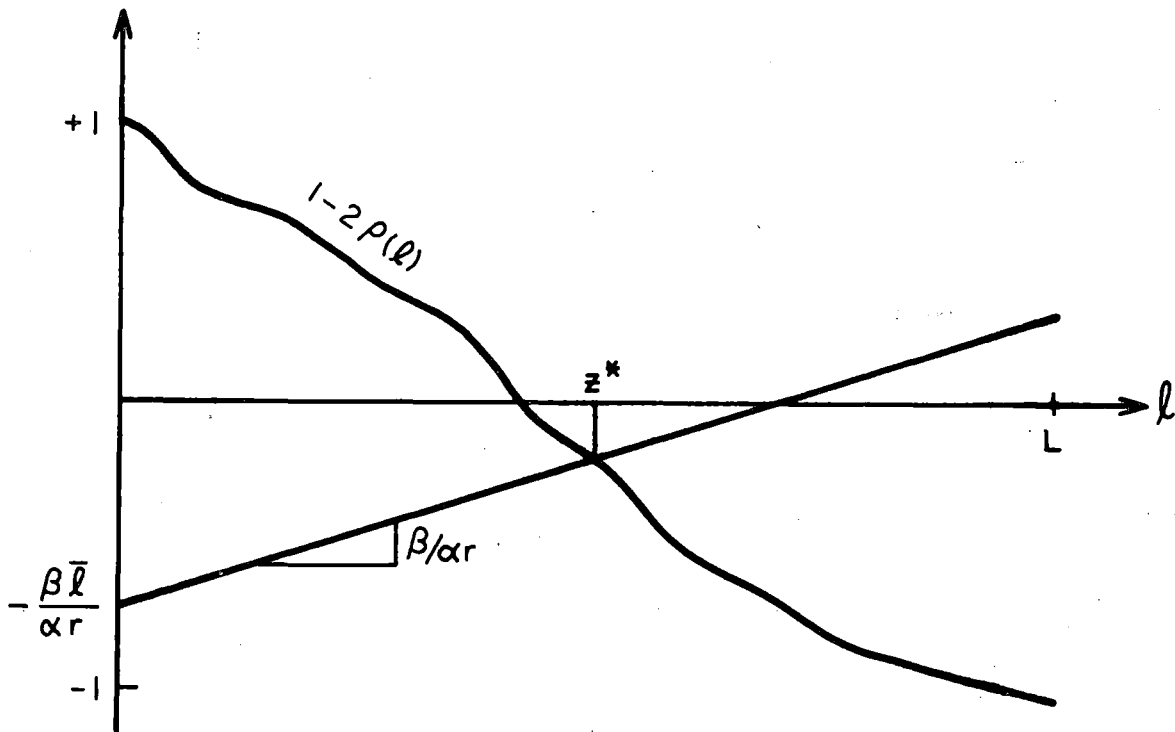


FIG. 1

The natural fulfillment of the necessary condition for a minimum is evident in the graph. Since α , β , and r are positive, the straight line has a positive slope, the curve of $1 - 2P(\ell)$ has a slope of $-2dP(\ell)$ which is negative, so the two curves can intersect at only one point. If $\beta = 0$, the straight line becomes the abscissa, and z^* occurs when $P(\ell) = \frac{1}{2}$. If there is a range of values $a \leq \ell \leq b$ such that $P(a) = P(b) = \frac{1}{2}$, then interval $[a, b]$ is the locus of optimal locations. One obvious limitation, that all parameters have finite values, insures that a solution of (10) always exists.

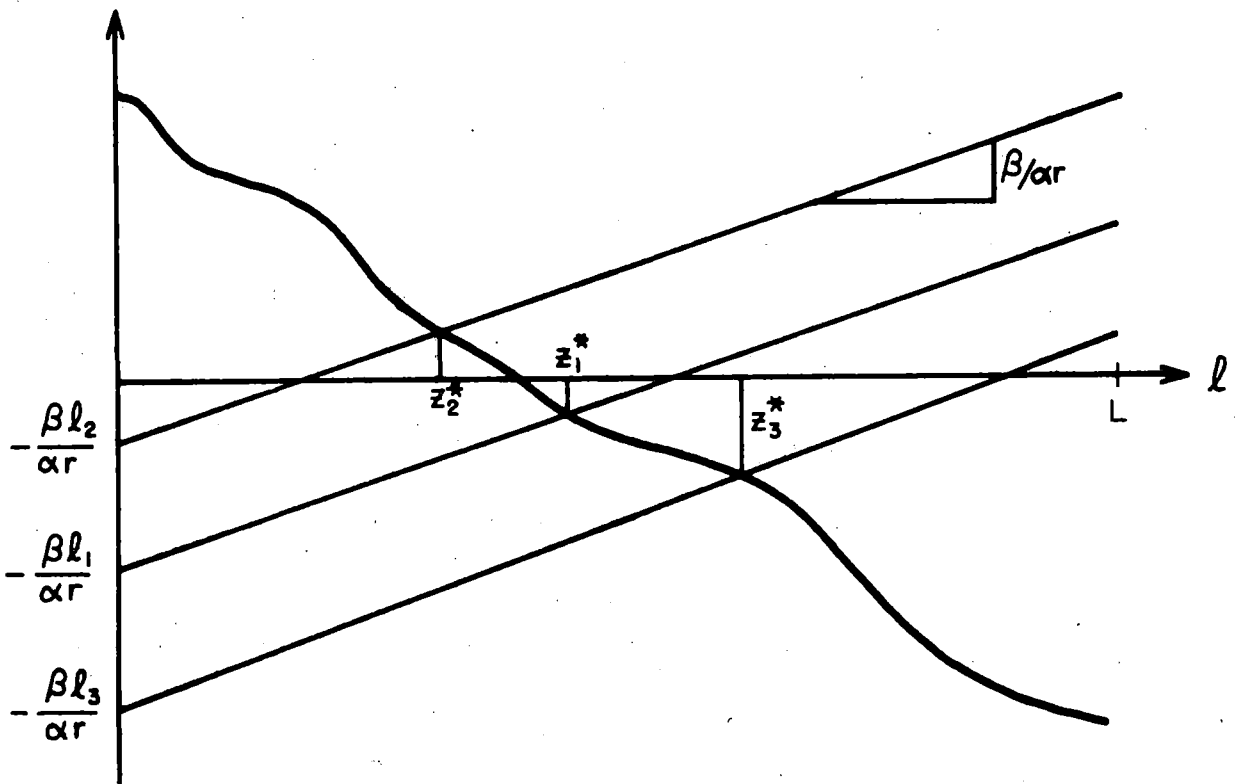


FIG. 2

In Figure 2 the effect of a change in $\bar{l}$ is shown. As one would expect, if the center of mass of the fire location distribution tends to the left ($\bar{l}$ decreases) z^* will tend toward the origin, and as $\bar{l}$ increases, z^* tends toward the right. It's interesting to notice that z^* depends only on the first moment of $P(l)$. This means that variation in the location of fires doesn't influence the optimal location of the fire fighting forces, and a few points at a tail-end of the distribution can greatly influence z^* . This runs counter to intuition, because a few slow growing fires in comparatively low-valued land should not have a great effect on z^* . Remembering that the model assumes G_A and C_B independent of l eliminates this difficulty, and points out the inadequacy of this simple model for practical results.

THEOREM: z^* is located between $\bar{l}$ and the median of $P(l)$.

PROOF: $1 - 2P(z^*) = \frac{\beta}{\alpha r}(z^* - \bar{l})$; $\beta, \alpha, r > 0$

$$z^* \leq m_l \Rightarrow 1 - 2P(z^*) \geq 0 \Rightarrow z^* \geq \bar{l}$$

$$z^* \geq m_l \Rightarrow 1 - 2P(z^*) \leq 0 \Rightarrow z^* \leq \bar{l}$$

$$\therefore \bar{l} \leq z^* \leq m_l \text{ or } m_l \leq z^* \leq \bar{l}$$

An obvious corollary is if $m_l = \bar{l}$, then $m_l = z^* = \bar{l}$.

III. Considering G_D and C_B as Functions of l

Since the terrain along a road varies among grass, rocks, timberland, etc., it is reasonable to assume that the potential fire growth rate and the value of the land burned will be a function of l . Introducing these conditions doesn't change the distance-dependent cost

equation (Eq. (8)), or the total expected cost equation (Eq. (9)), but the minimum of (9) with respect to z is affected by the G_D and C_B terms remaining inside the integral signs. The terms $G_D(l)$ and $C_B(l)$ will be used to represent the potential fire growth rate and cost of area burned functions respectively. More generality can be achieved by considering $H = H(l)$, which will increase the complexity of the algebra.

$$(11) \quad C(u, G_D(l), C_B(l)) = r + H \left[\frac{C_X}{E} + \sqrt{\frac{2C_S}{E}} \sqrt{C_B(l) + KC_B(l)} + \frac{1}{H} C_B(l) G_D(l) \right] \frac{u}{r} + \frac{HC_B(l)}{2} \left(\frac{u}{r}\right)^2.$$

Setting $\frac{\partial}{\partial z} C(u, G_D(l), G_D(l); z) = 0$ and rearranging terms,

$$(12) \quad 1 - 2P(z^*) = \frac{E}{C_X} \left\{ \frac{1}{r} (\overline{C_B} z^* - \overline{C_B} l) - \left(\sqrt{\frac{2C_S}{E}} \sqrt{\overline{C_B} + K\overline{C_B}} + \frac{1}{H} \overline{C_B} \overline{G_D} \right) + 2 \int_0^{z^*} \left[\sqrt{\frac{2C_S}{E}} \sqrt{C_B(l) + KC_B(l)} + \frac{1}{H} C_B(l) G_D(l) \right] dP(l) \right\}.$$

where,

$$\overline{C_B} = \int_0^L C_B(\ell) dP(\ell), \text{ the center of potential cost of burn}$$

$$\overline{C_B \ell} = \int_0^L \ell C_B(\ell) dP(\ell)$$

$$\sqrt{\overline{C_B}} = \int_0^L \sqrt{C_B(\ell)} dP(\ell)$$

$$\overline{C_B G_D} = \int_0^L C_B(\ell) G_D(\ell) dP(\ell) .$$

The point z^* is a unique global minimum if

$$\begin{aligned} \frac{\partial^2}{\partial z^2} C(u, G_D(\ell), G_B(\ell), z) &= 2P(z^*) + \frac{EC_B}{rC_X} \\ &+ \frac{2E}{C_X} \left[\sqrt{\frac{C_S}{2E}} \sqrt{C_B(z^*)} + KC_B(z^*) + \frac{1}{H} C_B(z^*) G_D(z^*) \right] P(z^*) > 0 . \end{aligned}$$

Using the physical argument that all parameters are non-negative and not identically zero, and all probabilities are non-negative, this condition is always satisfied. There is one pathological situation which invalidates this argument. A stretch of rocks and sand can be considered as having $p(\ell) = C_B(\ell) = G_D(\ell) = 0$; in this interval

$\frac{\partial^2}{\partial z^2} C(u, G_D(\ell), C_B(\ell); z) = 0$. This means that if z^* is contained in this interval, all points in the interval will satisfy Equation (12).

A graphical solution to Equation (12) is shown in Figure 3.

Curve (1) represents $\frac{E}{C_X r} (\bar{C}_B z - \bar{C}_B \ell)$, a straight line of slope $\bar{E}\bar{C}_B/C_X r$ and y-intercept $-\bar{E}\bar{C}_B \ell/C_X r$. Curve (2) is $-\frac{E}{C_X} (\sqrt{2\bar{C}_S/E} \sqrt{\bar{C}_B} + K\bar{C}_B + \frac{1}{H} \bar{C}_B \bar{G}_D)$, which is independent of z . Curve (3) is the integral portion of Equation (12), which may be decomposed into three simpler terms, a monotonically increasing function of z . The sum of these three curves is the right hand side of Equation (12), the heavy curve in Figure 3, and the intersection with $1 - 2P(z)$ yields the point z^* .

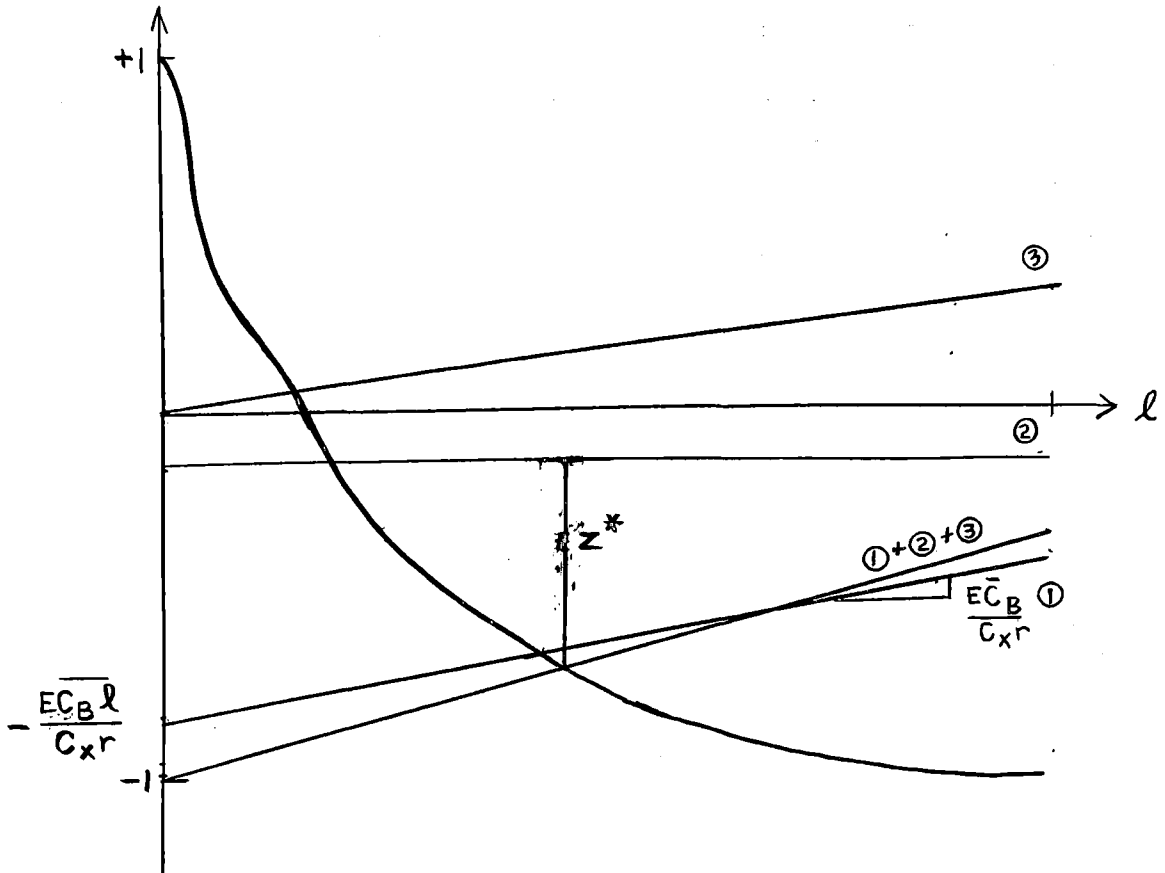


FIG. 3

Notice that this extension of the model overcomes the difficulty mentioned in Section II. The point z^* is dependent upon the form of $P(l)$, and the functions $C_B(l)$ and $G_D(l)$, as well as moments of these functions.

Equation (11) is simplified if $G_D(l) = G_D$ or $C_B(l) = C_B$, the situation where both $G_D(l)$ and $C_B(l)$ are constants is the model of Section II. The following results are easily obtainable from Equation (12).

If $C_B(l) = C_B$, then

$$(12a) \quad 1 - 2P(z^*) = \frac{C_B H}{a r} (z^* - \bar{l}) + \frac{C_B}{a} \left(2 \int_0^{z^*} G_D(l) dP(l) - \bar{G}_D \right),$$

$$\text{where } a = C_S H \sqrt{2C_B/C_S E} + \frac{C_X H}{E} + C_B H K.$$

If $G_D(l) = G_D$, then

$$(12b) \quad 1 - 2P(z^*) = \frac{H}{r a} (\bar{C}_B z^* - \bar{C}_B \bar{l}) + \frac{b}{a} \left[2 \int_0^{z^*} \sqrt{C_B(l)} dP(l) - \sqrt{\bar{C}_B} \right] \\ + \frac{d}{a} \left[2 \int_0^{z^*} C_B(l) dP(l) - \bar{C}_B \right],$$

$$\text{where } a = \frac{C_X H}{E}, \quad b = H \sqrt{2C_B/C_S E}, \text{ and } d = H K + G_D.$$

In many actual cases, it's been found that the acceleration component of fire growth, H , is zero, or very close to zero. If $H = 0$ in Equation (11), then Equation (12) is reduced to

$$(12c) \quad \int_0^{z^*} C_B(l)G_D(l)dP(l) = \int_{z^*}^L C_B(l)G_D(l)dP(l) ,$$

or

$$(12d) \quad 2 \int_0^{z^*} C_B(l)G_D(l)dP(l) = \overline{G_D C_B} .$$

Equation (12c) shows that z^* is the median of the function $f(l) = C_B(l)G_D(l)p(l)$. Equation (12d) provides a straightforward graphical solution of Equation (12c), which is illustrated in Figure 4.

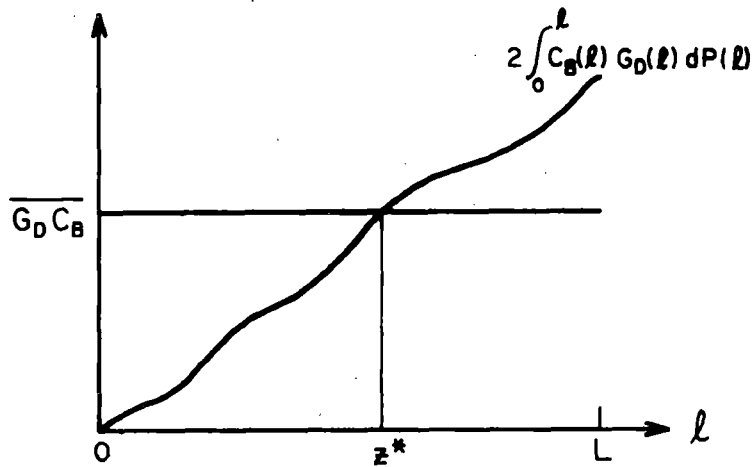


FIG. 4

IV. A Numerical Example

In any real forest, fuel and fire danger conditions will be constant over short intervals of distance along a road, so that the functions $P(l)$, $C_B(l)$, and $G_D(l)$ are represented by step functions. This means that the integrals in Equation (12) will be evaluated as sums. For clarity of presentation, the distance intervals in this example will be identical; this is not expected to be the case in actual applications.

Let us consider a 10-mile stretch of road, where:

$$\begin{aligned}
P(l) &= \frac{3l}{50} & C_B(l) &= 100 \text{ \$/acre} & G_D(l) &= 3 \frac{\text{acres}}{\text{hr.}} & 0 \leq l \leq 5 \\
P(l) &= \frac{3}{10} + \frac{l-5}{6} & C_B(l) &= 400 \text{ \$/acre} & G_D(l) &= 6 \frac{\text{acres}}{\text{hr.}} & 5 < l \leq 8 \\
P(l) &= \frac{8}{10} + \frac{l-8}{10} & C_B(l) &= 900 \text{ \$/acre} & G_D(l) &= 2 \frac{\text{acres}}{\text{hr.}} & 8 < l \leq 10
\end{aligned}$$

$$C_X = 3 \text{ \$/man-hour} \quad C_S = 50 \text{ \$/man}$$

$$E = 0.1 \text{ acres/hr}^2 - \text{man} \quad K = 0.03 \text{ hr.}$$

$$H = 3.0 \text{ acres/hr}^2 \quad r = 30 \text{ mph}$$

Then,

$$\bar{C}_B = 100 (0.3) + 400 (0.5) + 900 (0.2) = 410 \text{ \$/acre}$$

$$\bar{C}_B l = 5 \times 100 (0.3) + 3 \times 400 (0.5) + 2 \times 900 (0.2) = 1110 \frac{\text{\$ mile}}{\text{acre}}$$

$$\sqrt{\bar{C}_B} = 10 (0.3) + 20 (0.5) + 30 (0.2) = 19 (\text{dollar})^{1/2} / (\text{acre})^{1/2}$$

$$\bar{C}_B G_D = 3 \times 100 (0.3) + 6 \times 400 (0.5) + 2 \times 900 (0.2) = 1650 \text{ \$/hr.}$$

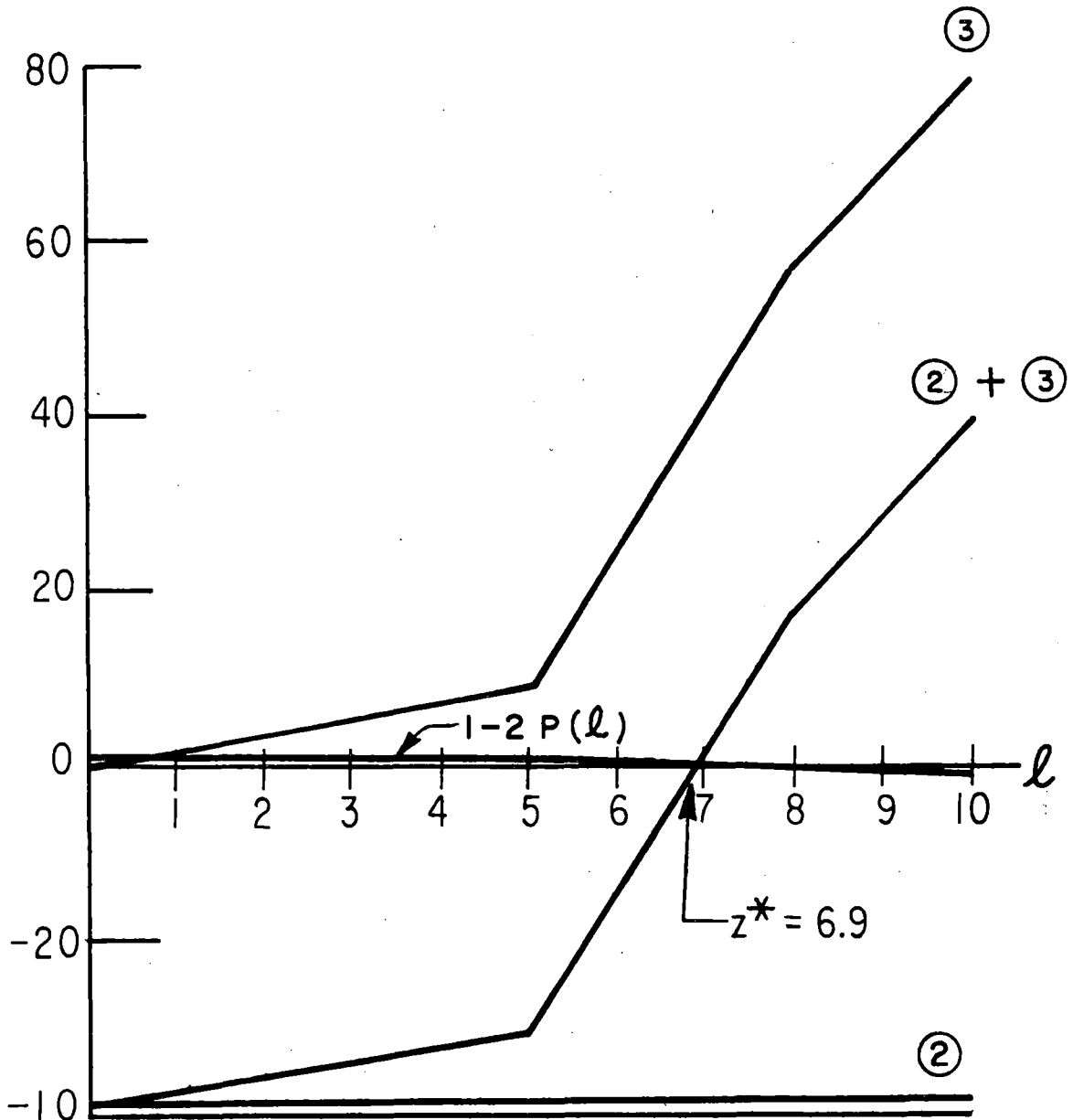
$$\frac{E \bar{C}_B}{C_X r} = \frac{41}{90} = .455$$

$$- \frac{E \bar{C}_B l}{C_X r} = - \frac{111}{90} = -1.233$$

$$- \frac{E}{C_X} \left(\sqrt{\frac{2C_S}{E}} \sqrt{\bar{C}_B} + K \bar{C}_B + \frac{1}{H} \bar{C}_B G_D \right) = - \frac{1162.7}{30} = -38.76$$

$$\frac{2E}{C_X} \int_0^l [15.8 \sqrt{C_B(l)} + .03 C_B(l) + \frac{1}{3} C_B(l) G_D(l)] dP(l) = < \begin{cases} 1.68l & 0 \leq l \leq 5 \\ 16.18l - 72.5 & 5 < l \leq 8 \\ 10.58l - 27.64 & 8 < l \leq 10 \end{cases}$$

Referring to Fig. 3, curve ① is negligible compared to curve ③, and will be dropped from consideration. The following graph is constructed in a straightforward manner from the preceding results.



Suppose a fire is spotted three miles to the left of z^* . Then $C_B = 100$, $G_D = 3$, and from Equation (29),

$$x^* = 30 + 3.09 \sqrt{10} + 3/10 \sqrt{10} = 40.7 \approx 41.$$

If the fire was located three miles to the right of z^* , then $C_B = 900$, $G_D = 2$, and

$$x^* = 30 + 2.09 \sqrt{90} + 3/10 \sqrt{90} = 52.7 \approx 53.$$

V. Rate of Travel Considered as a Function of l

If the roads through the forest contains long steep grades, the assumption that vehicular rate of travel is a constant is unrealistic. This difficulty cannot be handled by further modification of the distance between two points as expressed in "constructive miles" because hills produce the effect that the time to travel from a to b is not equal to the time to travel from b to a.

This condition means that r cannot be expressed as a function of l , because r depends on which direction the interval $[l, l + dl]$ is approached; i.e., r is double-valued on l . This difficulty can be overcome by making r a function of l and z in the following manner:

$$r(l, z) = r_L(l) \quad 0 \leq l < z$$

$$r(l, z) = r_R(l) \quad z < l \leq L$$

$$r(l, z) = 0 \quad l = z$$

where $r_L(l)$ is the rate of travel towards z from the left, and $r_R(l)$ is the rate of travel towards z from the right.

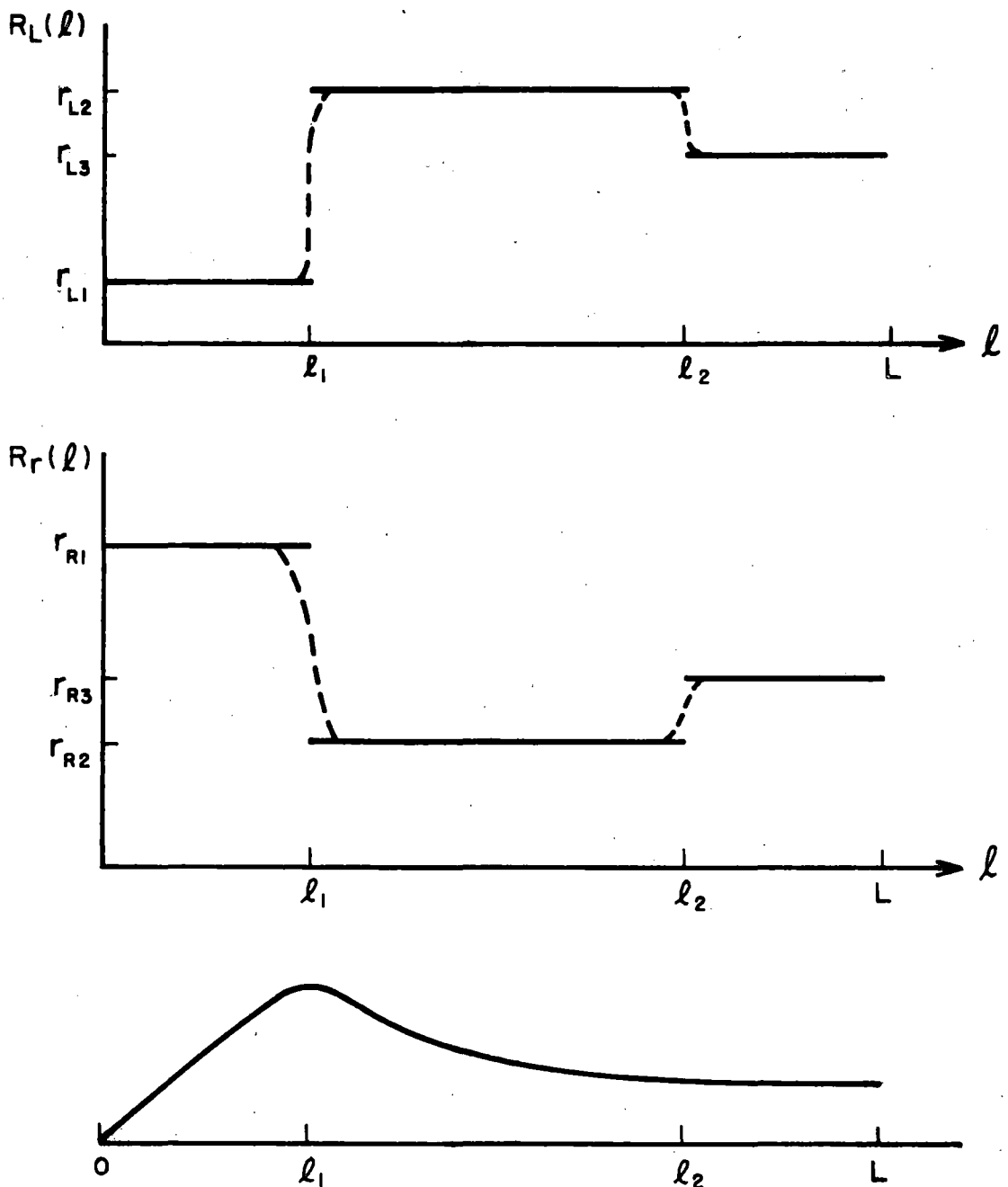


FIG. 5

Figure 5 shows a typical rate of travel graph. The interval $[0, l_1)$ represents the road sloping upward from the left, $[l_1, l_2)$ is a downhill slope from left to right that is not as steep as the previous uphill slope, and $[l_2, L]$ is a level road segment. The dotted lines represent the real situation where the grade

changes, smoothly at the peak of the hill; the assumption of a step function makes computation easier and shouldn't introduce noticeable errors.

The total cost equation to be minimized is

$$C(u; z) = \int_0^z \left[r + \frac{\alpha(z-l)}{r_L(l)} + \frac{\beta(z-l)^2}{2r_L(l)^2} \right] dP(l) \\ + \int_z^L \left[r + \frac{\alpha(l-z)}{r_R(l)} + \frac{\beta(l-z)^2}{2r_R(l)^2} \right] dP(l).$$

The necessary condition for a minimum $\frac{dC(u; z)}{dz} = 0$ yields

$$(13) \quad \int_0^{z^*} \left[\frac{\alpha}{r_L(l)} + \frac{\beta(z^*-l)}{r_L(l)^2} \right] dP(l) \\ = \int_{z^*}^L \left[\frac{\alpha}{r_R(l)} - \frac{\beta(z^*-l)}{r_R(l)^2} \right] dP(l).$$

Convexity is shown if

$$\frac{d^2 C(u; z)}{dz^2} = \int_0^{z^*} \frac{\beta}{r_L(l)^2} dP(l) + \left(\frac{\alpha}{r_L(l)} + \frac{\alpha}{r_R(l)} \right) P(z^*) \\ + \int_{z^*}^L \frac{\beta}{r_R(l)^2} dP(l) > 0.$$

Since each of the terms is positive, the inequality always holds and (13) always gives a unique z^* such that $C(u; z^*)$ is a minimum.

The solution, z^* , of Equation (13) is found by parametric methods because the value of z determines the functions $r_L(l)$ and $r_R(l)$,

which are step functions with steps at $0, l_1, l_2, \dots, l_1, \dots, l_n = L$.

Letting r_{L_i} and r_{R_i} be the rates of travel when $l_{i-1} < l \leq l_i$ and assuming $l_{k-1} < z^* \leq l_k$, where l_i represents the "jump" points of $r(l)$ and k is a specific value of i , we can write (13) as:

$$\begin{aligned}
 (14) \quad & \sum_{i=1}^{k-1} \int_{l_{i-1}}^{l_i} \left[\frac{\alpha}{r_{L_i}} + \frac{\beta(z^* - l)}{r_{L_i}^2} \right] dP(l) \\
 & + \int_{l_{k-1}}^{z^*} \left[\frac{\alpha}{r_{L_k}} + \frac{\beta(z^* - l)}{r_{L_k}^2} \right] dP(l) \\
 & = \int_{z^*}^{l_k} \left[\frac{\alpha}{r_{R_k}} + \frac{\beta(z^* - l)}{r_{R_k}^2} \right] dP(l) \\
 & + \sum_{i=k+1}^n \int_{l_{i-1}}^{l_i} \left[\frac{\alpha}{r_{R_k}} - \frac{\beta(z^* - l)}{r_{R_k}^2} \right] dP(l) ,
 \end{aligned}$$

where $k = 1, 2, \dots, n$ and $\sum_{k=1}^0 (0) = \sum_{i=n+1}^n (0) \equiv 0$.

Notice that the terms on the left hand side of (14) increase monotonically with k , and the terms on the right hand side decrease monotonically with k . If we assume $z^* = l_{k-1}$, and actually $z^* > l_{k-1}$, then the left hand side will be smaller than the right hand side; if we assume $z^* = l_k$, and actually $z^* \leq l_k$, the left hand side will be $\geq$ the right hand side. By letting $k = 1, 2, 3, \dots, n$, we can

find the interval containing z^* by getting the smallest k such that the right hand side of (14) is greater than the left hand side, and $z_{k-1} < z^* \leq z_k$.

The bounds of z^* can be tightened by dividing the interval $(l_{k-1}, l_k]$ into m equal subintervals, letting $z^* = l_{k-1} + p \frac{l_k - l_{k-1}}{m}$, $p=1, 2, \dots, m$, and looking for the first sign change when the right hand side is subtracted from the left hand side.

When z^* is calculated in this manner, unless z^* is at the end point of one of the intervals, the answer will be z^0 , an estimate of z^* which, is as close to z^* as desired. The maximum difference between z^0 and z^* is $\Delta = \max \{z^0 - l_{k-1}, l_k - z^0\}$, and the difference between the two sides of (14) is $\frac{\partial C(u, z)}{\partial z} \Big|_{z=z^0} = \delta$. Hence,

$$(15) \quad C(u; z^*) - C(u; z^0) \leq d \cdot \Delta.$$

From (15) the allowable error of z^0 as an estimate of z^* can be calculated in terms of cost.

Figure 6 is a block diagram of this algorithm.

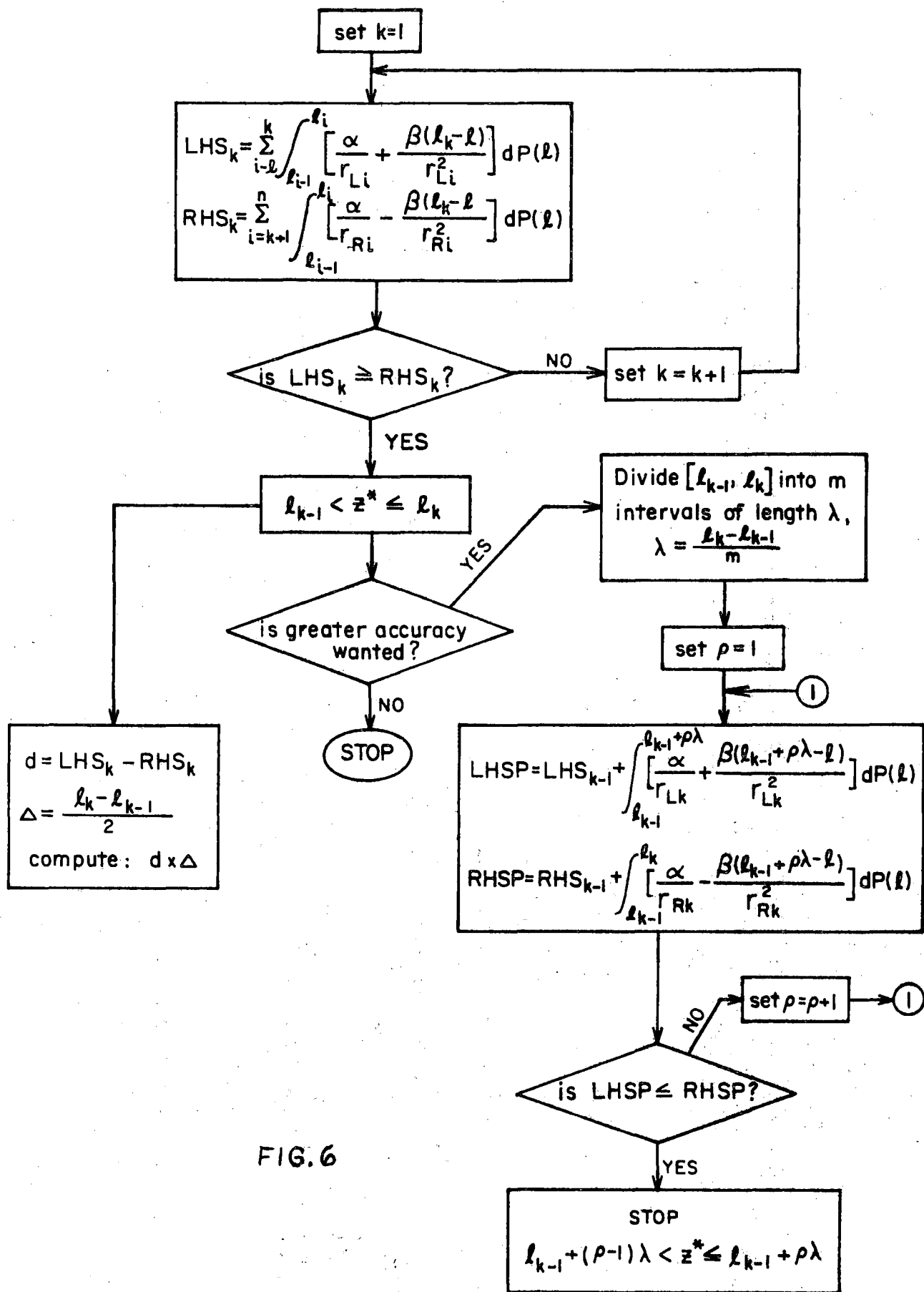


FIG. 6

Since the downhill rate of travel is less than the uphill rate of travel, z^* tends toward peaks. The interval $[\ell_{k-1}, \ell_k]$ has a valley and a peak at its endpoints, and it has been conjectured that z^* will always occur at endpoint that is a peak; this is not true! Although z^* is being "pulled" up the hill by $r(\ell)$, $P(\ell)$ may "pull" z^* down the hill toward the side of the forest with the greater expected fire damage. If the components of α and β , C_B and G_d , are considered as functions of ℓ , it becomes quite clear that one parameter will not determine z^* unless extreme values are assumed for all the parameters.

VI. The Optimum Number of Men to Send to a Fire

Once a fire has been detected, its location, growth rate, and the value of the burning area are known. We have established that

$$\begin{aligned}
 C^* = & \{C_F + C_S [(G_D(\ell) + HK) \sqrt{\frac{2C_B(\ell)}{C_S}} + \frac{H}{E}] + \frac{C_X}{E} [G_D(\ell) + HK] \\
 & + C_B(\ell) [KG_D(\ell) + y_D + \frac{1}{2} HK^2] \} \\
 & + \{C_S^H \sqrt{\frac{2C_B(\ell)}{C_S^E}} + \frac{C_X^H}{E} + C_B(\ell) [G_D(\ell) + HK]\} \left[\frac{u}{r(\ell)} \right] \\
 & + \left\{ \frac{HC_B(\ell)}{2} \right\} \left[\frac{u}{r(\ell)} \right]^2
 \end{aligned}$$

and,

$$(2a) \quad x^* = \frac{H}{E} + (G_D(\ell) + HK) \sqrt{\frac{C_B(\ell)}{2C_S^E}} + \left(\frac{H}{r(\ell)} \right) \frac{C_B(\ell)}{2C_S^E} u .$$

If $G_D(l)$, $G_B(l)$ and $r(l)$ are constants, x^* is a linearly increasing function of u , otherwise no general relationships between x^* and u exist. The above equation suggests construction of a table showing the average strength of the optimum force to send to homogeneous intervals, i.e., constant C_B , G_D , and r over the interval, centered u^0 miles from z^* .

The value of K is the only parameter that can be controlled. The changes in x^* and c^* due to a change in K are given by

$$\Delta x^* = H \sqrt{C_B(l)/2C_S E} \Delta K$$

$$\Delta c^* = \left\{ H \left[\frac{C}{E} + \sqrt{2C_S C_B(l)/E} + \frac{u C_B(l)}{r(l)} + K C_B(l) \right] + C_B(l) G_D(l) \right\} \Delta K$$

When H is small, the change in x^* is small; but the change in c^* is predominately due to the $C_B(l) \cdot G_D(l)$ term which may be significant. The change in c^* due to a change in $r(l)$ is

$$\Delta c^* = -u \left\{ \frac{H}{r(l)^2} \left[\frac{1}{r(l)} + \frac{C}{E} + \sqrt{2C_S C_B(l)/E} \right] + C_B(l) G_D(l) \right\} \Delta r(l)$$

which is also dominated by $C_B(l) \cdot G_D(l)$ for small H .

If there is choice of suppression forces, where one force has a longer mobilization time but a faster rate of travel, this force is preferable if

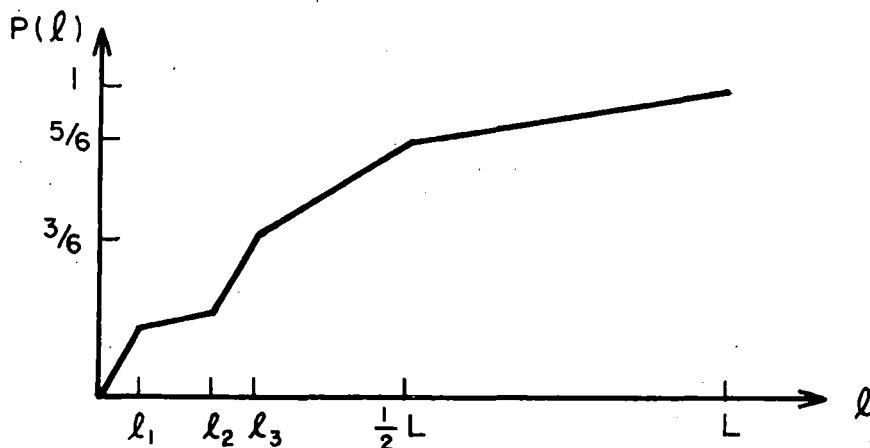
$$\frac{\text{Decrease in } C^* \text{ due to } r(l)}{\text{Increase in } C^* \text{ due to } K} < 1.$$

For $H \neq 0$, this criterion reduces to $\frac{\Delta r(l)}{\Delta K} < \frac{r(l)^2}{u}$, and it holds for all values of all the fire cost parameters, and any location of fire and suppression force. From this relationship we see that as u increases and r decreases, the time saved by additional rate of travel is more important than the time lost by the greater mobilization cost.

VII. The n-Station Location Problem

It's obvious that several (n) fire suppression equipment stations are needed to protect a large area. The procedure to be used is to solve the 2-station problem and to show the natural extension to n-stations.

The first step is to determine d , the demarcation point between the area of coverage of the two stations. This is not an easy problem, as the following example will illustrate.



This $P(l)$ represents a road with two highly inflammable stretches, $(0, l_1)$ and (l_2, l_3) , at one end. Two possible decision rules for determining d are:

(a) give each station half the fires, and (b) give each station half the road. If rule (a) is used, one station will have to cover three-fourths of the road which will involve large travel times to some fires. If rule (b) is used, one station will get five-sixths of the fires--which means either it's overloaded or the other station is underloaded. When the effects of $C_B(l)$, $G_D(l)$, and $r(l)$ are

added, the calculation of d becomes more complex.

The total expected cost equation for two locations is

$$(16) \quad C(u; z_1, z_2, d) = \int_0^{z_1} C(z_1 - l) dP(l) + \int_{z_1}^d C(l - z_1) dP(l) \\ + \int_d^{z_2} C(z_2 - l) dP(l) + \int_{z_2}^L C(l - z_2) dP(l) .$$

Minimizing (16) with respect to z_1 , z_2 , and d will give the optimal suppression equipment locations and the demarcation point between the area of coverage of the two stations. Thus, we want to find z_1^* , z_2^* , and d^* satisfying

$$\frac{\partial}{\partial d} C(u; z_1, z_2, d) = 0 = C(d^* - z_1^*) = C(z_2^* - d^*)$$

$$\frac{\partial}{\partial z_1} C(u; z_1, z_2, d) = 0 = \int_0^{z_1^*} \frac{\partial C(z_1 - l)}{\partial z_1} dP(l) + \int_{z_1^*}^{d^*} \frac{\partial C(l - z_1)}{\partial z_1} dP(l)$$

$$\frac{\partial}{\partial z_2} C(u; z_1, z_2, d) = 0 = \int_{d^*}^{z_2^*} \frac{\partial C(z_2 - l)}{\partial z_2} dP(l) + \int_{z_2^*}^L \frac{\partial C(l - z_2)}{\partial z_2} dP(l)$$

These three independent simultaneous equations can be solved on a computer; and a graphical solution for the case $C(u) = r + \alpha u$ is given in the appendix.

This analysis can be extended to n locations by simultaneously minimizing $C(u; z_1, \dots, z_n, d_1, \dots, d_{n-1})$ with respect to each of the variables. This results in a system of $2n-1$ equations which will not be easy to solve. In addition to the computational difficulties encountered in this method, the author feels that several important factors influencing the selection of d are not incorporated in the present

model. The choice of a demarcation point might depend on the strength of the suppression forces assigned to each station, the composition of these forces and the variations in the terrain between the two stations. In short, research into the logistics of fire suppression is needed before optimum demarcation points, if they exist, can be calculated. In the ensuing discussion, d is assumed to be given.

Station 1 will be responsible for $0 \leq l \leq d$, and Station 2 for $d \leq l \leq L$. The stations will have the "renormalized" fire probability density functions $f_1(l)$ and $f_2(l)$ respectively, where:

$$f_1(l) = \frac{p(l)}{\int_0^d p(l) dl} = \frac{p(l)}{P(d)} \quad 0 \leq l \leq d$$

$$f_2(l) = \frac{p(l)}{\int_d^L p(l) dl} = \frac{p(l)}{P^c(l)} \quad d \leq l \leq L$$

$$F_1(l) = \int_0^l f_1(x) dx, \quad F_2(l) = \int_d^l f_2(x) dx \quad \text{have the properties:}$$

$$F_1(0) = 0, \quad F_1(d) = 1, \quad F_2(d) = 0, \quad F_2(L) = 1.$$

The total cost equation is

$$C(u, z_1, z_2) = \int_0^{z_1} C(z_1 - l) f_1(l) dl + \int_{z_1}^d C(l - z_1) f_1(l) dl \\ + \int_d^{z_2} C(z_2 - l) f_2(l) dl + \int_{z_2}^L C(l - z_2) f_2(l) dl$$

Solving for z_1^* :

$$\begin{aligned} \frac{\partial \mathcal{C}(u, z_1, z_2)}{\partial z_1} &= \int_0^{z_1} \left[\frac{\alpha}{r} + \frac{\beta(z_1 - l)}{r^2} \right] f_1(l) dl \\ &\quad - \int_{z_1}^d \left[\frac{\alpha}{r} + \frac{\beta(z_1 - l)}{r^2} \right] f_1(l) dl \\ &= \frac{2\alpha}{r} F_1(z_1) - \frac{\beta}{r^2} \int_0^d l f_1(l) dl + \frac{\beta z_1}{r^2} - \frac{\alpha}{r} = 0 . \end{aligned}$$

Let $\int_0^d l f_1(l) dl = \hat{l}$, the mean of $f_1(l)$

$$(16) \quad 1 - 2F_1(z_1^*) = \frac{\beta}{\alpha r} (z_1^* - \hat{l}) .$$

Solving for z_2^* :

$$\begin{aligned} \frac{\partial \mathcal{C}(u, z_1, z_2)}{\partial z_2} &= \int_d^{z_2} \left[\frac{\alpha}{r} + \frac{\beta(z_2 - l)}{r^2} \right] f_2(l) dl \\ &\quad - \int_{z_2}^L \left[\frac{\alpha}{r} + \frac{\beta(z_2 - l)}{r^2} \right] f_2(l) dl = 0 . \end{aligned}$$

$$(17) \quad 1 - 2F_2(z_2^*) = \frac{\beta}{\alpha r} (z_2^* - \tilde{l}) , \text{ where}$$

$$\tilde{l} = \int_d^L l f_2(l) dl , \text{ the mean of } f_2(l) .$$

Equations (16) and (17) are of the same form as Equation (10), so the solution of the 2-location problem is two optimal single locations.

Extending this analysis to n locations, we use the demarcation points d_i , and the "renormalized" probability density functions f_i to get

$C(u, z_1, z_2, \dots, z_n)$ equals

$$\begin{aligned} & \int_0^{z_1} C(z_1 - l) f_1(l) dl + \int_{z_1}^{d_1} C(l - z_1) f_1(l) dl \\ & + \sum_{i=2}^{n-1} \left[\int_{d_{i-1}}^{z_i} C(z_i - l) f_i(l) dl + \int_{z_i}^{d_{i+1}} C(l - z_i) f_i(l) dl \right] \\ & + \int_{d_{n-1}}^{z_n} C(z_n - l) f_n(l) dl + \int_{z_n}^L C(l - z_n) f_n(l) dl. \end{aligned}$$

By comparing the above expression with equation (9), it is clear that

$C(u, z_1, z_2, \dots, z_n)$ is a convex function with respect to each z_i , thus, its minimum occurs when

there. This will occur when

$$\frac{\partial}{\partial z_i} C(u, z_1, z_2, \dots, z_n) = 0 \quad \forall i$$

giving

$$(18) \quad 1 - 2F_i(z_i^*) = \frac{\beta}{\alpha r} (z_i^* - \bar{l}_i),$$

where

$$\bar{l}_i = \int_{d_{i-1}}^{d_i} l f_i(l) dl, \quad d_0 = 0, \text{ and } d_n = L.$$

Thus, the solution to the n-location problem is n solutions of a single location problem, and the analysis of the basic cost-dependent model extends to the n-location situation.

VIII. Use of the Model

Although the model considers only fires along a road, it is applicable as a decision making tool to the general location problem. An analysis of the incidence of fire in the Plumas National Forest shows that many of the fires that occurred during a ten-year period started on the main highway through the forest. Furthermore, forest personnel stated that the usual manner of traveling to a fire off the road is to drive down this highway as far as possible, and then go into the forest. Under these conditions, it is meaningful to minimize the time spent traveling along the road, and to consider these fires in calculating $P(l)$; the time spent on interior roads being considered an uncontrollable time delay. It is apparent that airborne methods of transportation invalidates this argument, but such methods are not in general use. Perhaps this model and a suitable extension could be used in comparing present transportation methods and the proposed airborne methods in feasibility studies.

One of the major difficulties in applying a mathematical model to a physical situation is the measurement of parameters. In this model, $C_B(l)$ and $G_D(l)$ pose the greatest measurement problems. In both cases, classification of land into categories such as low, low-intermediate, high-intermediate, and high, with appropriate numerical representations, should suffice. This will give rise to step functions, and the integrals in the formulas will have to be evaluated piece-wise,

as was done for $r(\ell)$. The simplifying assumption of a step-function seems appropriate, since forest lands are fairly uniform over specified areas, and the simplicity of the model makes error analysis quite straightforward.

REFERENCES

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Appendix: The Calculation of d^* when $C(u) = \gamma + \alpha u$

The necessary condition for a minimum of $C(u, z_1, z_2, d)$ is that the simultaneous equations

$$(1) \quad C(d^* - z_1^*) = C(z_2^* - d^*)$$

$$(2) \quad \int_0^{z_1^*} \frac{\partial C(z_1 - l)}{\partial z_1} dP(l) + \int_{z_1^*}^{d^*} \frac{\partial C(l - z_1)}{\partial z_1} dP(l) = 0$$

$$(3) \quad \int_{d^*}^{z_2^*} \frac{\partial C(z_2 - l)}{\partial z_2} dP(l) + \int_{z_2^*}^L \frac{\partial C(l - z_1)}{\partial z_1} dP(l) = 0$$

be satisfied. If $C(u) = \gamma + \alpha u$, equations (1), (2), and (3) become:

$$(1a) \quad d^* = (z_1^* + z_2^*)/2$$

$$(2a) \quad 2P(z_1^*) = P(d^*)$$

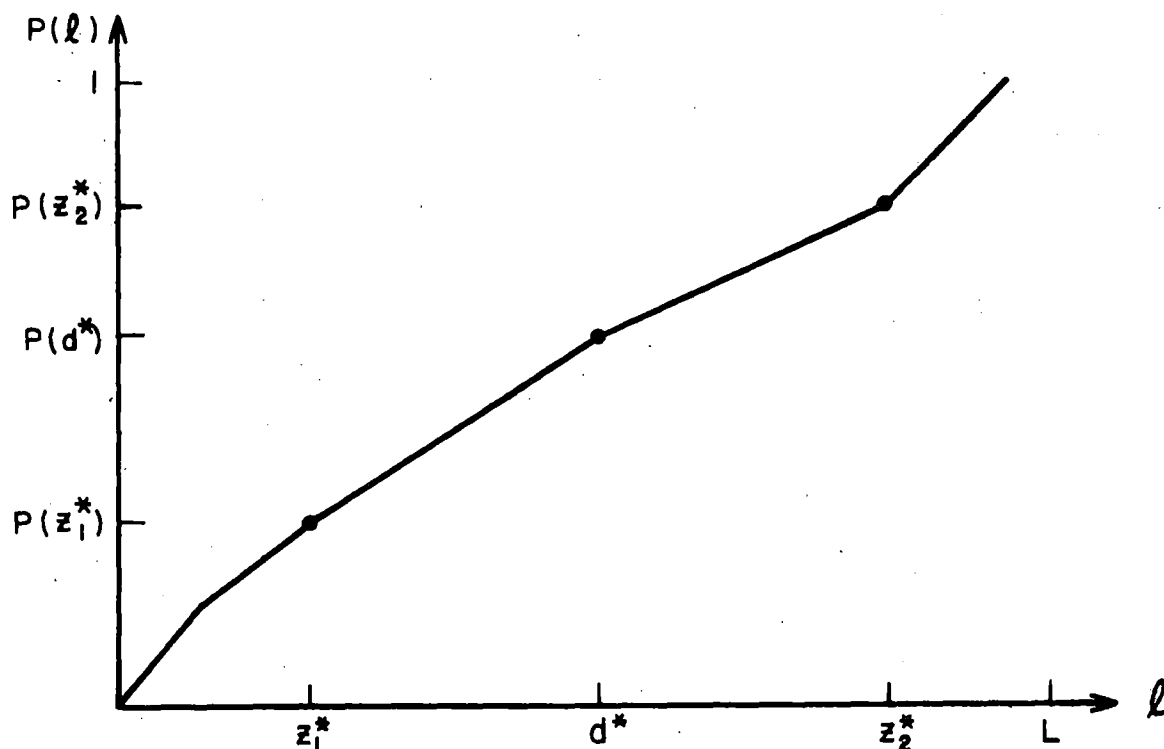
$$(3a) \quad 2P(z_2^*) = P(d^*) + 1.$$

Combining equations (2a) and (3a) we get:

$$(3b) \quad P(z_2^*) = P(z_1^*) + \frac{1}{2}.$$

Equations (1a), (2a), and (3b) are used for the graphical solution.

The solution technique is straightforward; one should keep in mind that the first equation relates to the abscissa and the other two are ordinate restrictions.



Notice that each station did not get 50% of the fires, but each station is located at the median of its renormalized fire probability density function. This result satisfies equation (18) and equation (10) for the two one-location problems.